An Enumerative Method for Encoding Spectrum Shaped Binary Run-Length Constrained Sequences

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***Abstract*—A method for encoding and decoding spectrum shaped binary run-length constrained sequences is described. The binary sequences with predefined range of exponential sums are introduced. On the base of Cover's enumerative scheme, recurrence relations for calculating the number of these sequences are derived. Implementation of encoding and decoding procedures is also shown.**

I. Introduction

Binary sequences with constrained run length of zeros are known in literature as dk sequences [1], [2]. In these sequences, single ones are separated by at least d, but not more then k zeros. A dkr sequence is a dk sequence, ending in a run of not more then r trailing zeros. A $dklr$ sequence is a dkr sequence, beginning with a run of not more then l leading zeros. Among these, sequences with spectral null at zero frequency (dc) are notable. This implies a line encoding technique using NRZI rules.

By NRZI encoding [2] we understand mapping the source binary sequence $\boldsymbol{x} \in \{0,1\}^n$ to bipolar sequence $\boldsymbol{z} \in \{-1,1\}^n$ such that

$$z_j = \begin{cases} z_{j-1}, & x_j = 0, \\ -z_{j-1}, & x_j = 1, \end{cases}$$
$$z_0 = 1.$$

The discrete Fourier transform (DFT) of this sequence is defined as

$$z_m^* = \sum_{j=0}^{n-1} z_{j+1} e^{-2\pi i \frac{mj}{n}}, \quad m = 0, \ldots, n-1.$$

The usual spectrum shaping requirements are first, a zero-mean value for the transmitted data sequence, and second, small power content at low frequencies. This second requirement can be written as

$$|z_m^*| < M,$$

where M is some constant. These zero-mean value sequences are called DC-free RLL or DCRLL [3].

Although a study of spectral properties of channel sequences was performed by Nyquist [4] as early as in the late 20s; regular papers concerning this problem were appeared in the late 60s – early 70s with the contributions coming from Gorog [5], Franklin and Pierce [6], and some other authors. In 1984 Pierobon [7] proved that the finite running digital sum condition is a necessary and sufficient condition for zero mean and spectral density vanishing at zero frequency. Later, Marcus and Siegel [8] have expanded the concept of the running digital sum to each component of the DFT. This allows extension of the spectral null control from zero up to the Nyquist frequency.

Enumerative encoding and decoding methods for DC-free sequences were suggested by Norris and Bloomberg [9], Immink [10], Vasilev [11], Braun and Immink [3].

A method for enumerative encoding these sequences with predefined dc component of the DFT $z_0^* = \sum_{j=1}^{n} z_j$, which is usually called a digital sum or a charge of the sequence $\boldsymbol{z}$, was suggested in [12]. Observe that $z_0^* \in [-n, n]$, where z_0^* admits even values whenever n is even and odd values whenever n is odd.

II. Spectrum Shaping

Let $\{0,1\}^n$ be the set of all binary sequences of length n and let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ denote a generic element of this set. Let $\mathscr{S}(n) = \{\boldsymbol{x} \in \{0,1\}^n \mid$ satisfies the d,k,r constraints and $x_1 = 1\}$ and let $\hat{\mathscr{S}}(n) = \{\boldsymbol{x} \in \{0,1\}^n \mid$ satisfies the d,k,l,r constraints$\}$.

Let C_n^σ be the number of sequences from $\mathscr{S}(n)$; these sequences have charge $\sigma = z_0^*$. Using Cover's method [13], the number of these sequences can be computed, as is shown in [12], using recurrence relation

$$C_n^\sigma = \begin{cases} 0, & n < d+1, \\ \sum_{j=d+1}^{\min(n,k+1)} C_{n-j}^{-\sigma-j}, & d+1 \le n \end{cases} \tag{1}$$

with initial conditions

$$C_n^{-n} = \begin{cases} 1, & n \le r+1, \\ 0, & \text{otherwise.} \end{cases}$$

We might expand this method to $z_1^* \ldots z_{n-1}^*$ components of the DFT, but it most likely will not have meaning. Indeed, recall that the DFT is a one-to-one transformation. If we go from the single spectral component to the vector $\boldsymbol{z^*} = (z_0^*, z_1^*, \ldots, z_{n-1}^*)$, then for an exact value of $\boldsymbol{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1}) \in \mathbb{C}^n$ we see that $C_n^{\boldsymbol{\sigma}}$, can be either equal to 1 whenever $\boldsymbol{\sigma} = \boldsymbol{z^*}$ or equal to 0 otherwise. Therefore, we must consider the numbers $C_n^{\sigma_m}$ for which spectral components z_m^* lie in some area; let this area be a ring $\mathscr{A}_{\sigma_m,\rho_{1,m},\rho_{2,m}}$ centered at σ_m with inner radius $\rho_{1,m} \in \mathbb{R}$ and outer radius $\rho_{2,m} \in \mathbb{R}$; i.e. the ring $\mathscr{A}_{\sigma_m,\rho_{1,m},\rho_{2,m}}$ is a set $\mathscr{A}_{\sigma_m,\rho_{1,m},\rho_{2,m}} = \{\sigma_m : \rho_{1,m} \le |z_m^* - \sigma_m| \le \rho_{2,m}\}$, $z_m^* \in \mathbb{C}$. The choice of a ring may be useful in a special case when we need only the magnitude range and do not need

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TABLE I
ALL LEXICOGRAPHICALLY ORDERED $dklr$ SEQUENCES OF LENGTH $n=8$. CONSTRAINTS: $d=2$, $k=4$, $l=1$, $r=3$.

N[a]	x_1 / z_1	x_2 / z_2	x_3 / z_3	x_4 / z_4	x_5 / z_5	x_6 / z_6	x_7 / z_7	x_8 / z_8	z_0^*	z_1^*
0	0	1	0	0	0	0	1	0	-2	(3.41, 3.41)
	1	-1	-1	-1	-1	-1	1	1		
1	0	1	0	0	0	1	0	0	0	(2.00, 4.83)
	1	-1	-1	-1	-1	1	1	1		
2	0	1	0	0	1	0	0	0	2	(0.00, 4.83)
	1	-1	-1	-1	1	1	1	1		
3	0	1	0	0	1	0	0	1	0	(-1.41, 3.41)
	1	-1	-1	-1	1	1	1	-1		
4	1	0	0	0	0	1	0	0	-2	(0.00, 4.83)
	-1	-1	-1	-1	-1	1	1	1		
5	1	0	0	0	1	0	0	0	0	(-2.00, 4.83)
	-1	-1	-1	-1	1	1	1	1		
6	1	0	0	0	1	0	0	1	-2	(-3.41, 3.41)
	-1	-1	-1	-1	1	1	1	-1		
7	1	0	0	1	0	0	0	1	0	(-4.83, 2.00)
	-1	-1	-1	1	1	1	1	-1		
8	1	0	0	1	0	0	1	0	-2	(-4.83, 0.00)
	-1	-1	-1	1	1	1	-1	-1		

[a] By N we denote the lexicographic index of sequence.

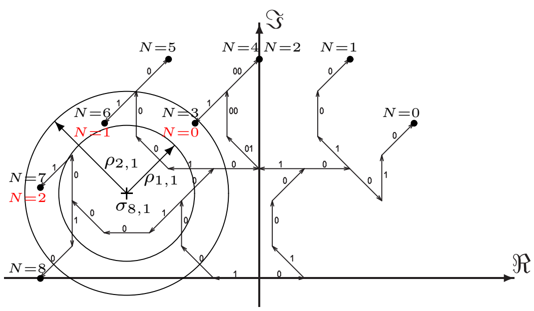

Fig. 1. Paths and spectral components z_1^* in the complex plane.

an argument. In this case centre of the ring should coincide with the origin of the complex plane and we put $\sigma_m = 0$.

Initially we consider the DFT of n-length sequences. Enumerative scheme implies recurrent calculation of the numbers of these sequences. From (1) it follows that each level of the recursion diminishes n. Therefore, we must consider two different types of lengths; first is the length of the sequence and is denoted by n; second are the lengths of the nested subsequences and are denoted by $\tilde{n}$. Moreover, let the spectral properties are given for a large sequence of length n. With l and r constraints and the shift theorem, we can obtain this n-length sequence by concatenating $K \in \mathbb{N}$ subsequences of length $\tilde{n}$ such that $n = K\tilde{n}$.

Also we intend neither to reject nor even to relax the run-length constraints for at least two reasons: first, these constraints bound the order of the recurrence relation (1) and second, exact values of d and k constraints should make synchronization control easier.

For example, Table I shows sequences $\boldsymbol{x}$, $\boldsymbol{z}$, spectral components z_0^* and z_1^* respectively. The spectral components z_1^* are depicted in Fig. 1. An example of the ring $\mathscr{A}_{\sigma_{8,1},\rho_{1,1},\rho_{2,1}}$ is superimposed on this image. We also show paths that lead from the original to z_1^*. Assuming that spectrum constraints for z_0^*, $z_2^* \dots z_7^*$ are relaxed, we summarize in Table II those sequences for which components z_1^* lie in the ring $\mathscr{A}_{\sigma_{8,1},\rho_{1,1},\rho_{2,1}}$.

TABLE II
ALL LEXICOGRAPHICALLY ORDERED SPECTRUM SHAPED SEQUENCES FROM TABLE I WITH $\sigma_{8,1} = (-2.93, 1.87)$, $\rho_{1,1} = 1.5$, $\rho_{2,1} = 2.25$.

N	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	z_1^*
0	0	1	0	0	1	0	0	1	(-1.41, 3.41)
1	1	0	0	0	1	0	0	1	(-3.41, 3.41)
2	1	0	0	1	0	0	0	1	(-4.83, 2.00)

III. THE NUMBER OF SEQUENCES

By $\omega_n = e^{\frac{2\pi i}{n}}$ denote a primitive nth root of unity. Now we shall give the following definition. Let $0 \le m \le n-1$, $\tilde{n} \le n$, $z \in \{-1, 1\}$, and let $\theta_m \in \mathbb{C}$ be some constant. Then an order j recurrence relation

$$\sigma_{\tilde{n},m} = z \sum_{t=0}^{j-1} \omega_n^{-mt} + \theta_m \sigma_{\tilde{n}-j,m}$$

is called a *trigonometric recurrence relation*. The constant θ_m is said to be a linear phase if $\theta_m = z\omega_n^{-mj}$.

This trigonometric recurrence relation is not basically very different from the Danielson-Lanczos identity [14]. There, an n-point DFT was split into the sum of two DFT; one is formed from the even-numbered points, the other from the odd-numbered points. Here, we add an exponential sum of the prefix to multiplying by the linear phase trigonometric recurrence relation of the subsequence.

Consider bipolar run-length constrained sequences of length $\tilde{n}$ and having an mth trigonometric recurrence relation ring $\mathscr{A}_{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}}$ centered at $\sigma_{\tilde{n},m}$ with inner radius $\rho_{1,m}$ and outer radius $\rho_{2,m}$; here the ring $\mathscr{A}_{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}}$ is a set $\mathscr{A}_{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}} = \{\sigma_{\tilde{n},m} : \rho_{1,m} \le |z_m^* - \sigma_{\tilde{n},m}| \le \rho_{2,m}\}$, $z_m^* \in \mathbb{C}$.

We compute the number of $\tilde{n}$-length subsequences of $\boldsymbol{z}$.

Let $C_{n,\tilde{n}}^{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}}$ be the number of these subsequences, which begin with one. Let $\widehat{C}_{n,\tilde{n}}^{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}}$ be the number of these sequences, which begin with a leading run of zeros.

Since an internal run of zeros succeeds a leading run of zeros, we see that the leading constraint l does not affect $C_{n,\tilde{n},m}^{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}}$. For convenience, below, under $C_{n,\tilde{n},m}^{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}}$ we imply $C_{n,\tilde{n},m}^{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}}(d,k,r)$ and under $\widehat{C}_{n,\tilde{n},m}^{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}}$ we similarly imply $\widehat{C}_{n,\tilde{n},m}^{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}}(d,k,l,r)$.

Proposition 1. *The numbers* $C_{n,\tilde{n},m}^{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}}$ *and* $\widehat{C}_{n,\tilde{n},m}^{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}}$ *can be obtained as:*

$$C_{n,\tilde{n},m}^{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}} = \begin{cases} a_{\tilde{n},m}, & \tilde{n} < d+1, \\ \sum_{j=d+1}^{\min(\tilde{n},k+1)} C_{n,\tilde{n}-j,m}^{\sigma_{\tilde{n}-j,m},\rho_{1,m},\rho_{2,m}} + b_{\tilde{n},m}, & d+1 \le \tilde{n}, \end{cases} \tag{2}$$

where

$$\sigma_{\tilde{n}-j,m} = -\omega_n^{mj}\left(\sigma_{\tilde{n},m} + \omega_n^{-m(j-1)/2}\frac{\sin\left(\frac{mj}{n}\pi\right)}{\sin\left(\frac{m}{n}\pi\right)}\right), \tag{3}$$

initial condition

$$a_{\tilde{n},m} = \begin{cases} 1, & \tilde{n} \le r+1 \text{ and } \rho_{1,m} \le |\sigma_{\tilde{n},m} - \tilde{\sigma}_{\tilde{n},m}| \le \rho_{2,m}, \\ 0, & \textit{otherwise}, \end{cases} \tag{4}$$

and additional condition

$$b_{\tilde{n},m} = \begin{cases} -1, & r+1 < \tilde{n} \le k+1 \\ & \text{and } \rho_{1,m} \le |\sigma_{\tilde{n},m} - \tilde{\sigma}_{\tilde{n},m}| \le \rho_{2,m}, \\ 1, & k+1 < \tilde{n} \le r+1 \\ & \text{and } \rho_{1,m} \le |\sigma_{\tilde{n},m} - \tilde{\sigma}_{\tilde{n},m}| \le \rho_{2,m}, \\ 0, & \textit{otherwise}, \end{cases} \tag{5}$$

where

$$\tilde{\sigma}_{\tilde{n},m} = -\omega_n^{-m(\tilde{n}-1)/2} \frac{\sin\left(\frac{m\tilde{n}}{n}\pi\right)}{\sin\left(\frac{m}{n}\pi\right)}. \tag{6}$$

If a leading series is running, then

$$\widehat{C}_{n,\tilde{n},m}^{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}} = \sum_{j=0}^{\min(\tilde{n},l)} C_{n,\tilde{n}-j}^{\hat{\sigma}_{\tilde{n}-j,m},\rho_{1,m},\rho_{2,m}} + \hat{b}_{\tilde{n},m},$$

where

$$\hat{\sigma}_{\tilde{n}-j,m} = \omega_n^{mj} \left(\sigma_{\tilde{n},m} - \omega_n^{-m(j-1)/2} \frac{\sin\left(\frac{mj}{n}\pi\right)}{\sin\left(\frac{m}{n}\pi\right)} \right), \tag{7}$$

additional condition

$$\hat{b}_{\tilde{n},m} = \begin{cases} -1, & r < \tilde{n} \le l \text{ and} \\ & \rho_{1,m} \le |\sigma_{\tilde{n},m} + \tilde{\sigma}_{\tilde{n},m}| \le \rho_{2,m}, \\ 0, & \textit{otherwise}. \end{cases} \tag{8}$$

Here $d \ge 0$, $k \ge d$, $l \ge 0$, $r \ge 0$.

Proof: First, consider initial and additional conditions (4) and (5). In the case of $\tilde{n} \le r+1$, there is only a trailing run in the subsequence. This trailing subsequence of $\boldsymbol{x}$ consists of one and $\tilde{n}-1$ zeros. The NRZI rule takes each term of the trailing subsequence to -1. It defines the mth exponential sum $\tilde{\sigma}_{\tilde{n},m}$, which corresponds to this subsequence, as follows:

$$\underbrace{\overbrace{\begin{matrix} 1 & 0 \ldots & 0 \\ -1 & -1 \ldots & -1 \end{matrix}}^{1 \le \tilde{n} \le r+1}}_{\tilde{\sigma}_{\tilde{n},m} = \sum_{t=0}^{\tilde{n}-1} (-1)\omega_n^{-mt}}.$$

Therefore, it gives us the only allowed sequence which length lies in the interval $[1, r+1]$ and $\tilde{\sigma}_{\tilde{n},m}$ lies in the ring $\mathscr{A}_{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}}$, i.e.

$$\rho_{1,m} \le |\sigma_{\tilde{n},m} - \tilde{\sigma}_{\tilde{n},m}| \le \rho_{2,m}. \tag{9}$$

Consider a finite geometric series; then recall the derivation of the Dirichlet kernel trigonometric identity and obtain

$$\sum_{t=0}^{\tilde{n}-1} \omega_n^{-mt} = \frac{1 - \omega_n^{-m\tilde{n}}}{1 - \omega_n^{-m}} \tag{10}$$

$$= \omega_n^{-m(\tilde{n}-1)/2} \frac{\sin\left(\frac{m\tilde{n}}{n}\pi\right)}{\sin\left(\frac{m}{n}\pi\right)}. \tag{11}$$

Multiplying (11) by -1, we obtain (6).

Now we must prove the recurrence relation in (2). According to Cover's enumerative method [13], we build the recursion by the following way. Let us consider a possible run of zeros, which follows the leading one, as a prefix for the following subsequences beginning also with one. Assuming the length j of the prefix grows from $d+1$ to $\min(\tilde{n}, k+1)$ and weight of this prefix equals one, we can consider a concatenation of the prefix and the following subsequences as

$$\underbrace{\overbrace{\begin{matrix} 1 & 0 \ldots & 0 \\ -1 & -1 \ldots & -1 \end{matrix}}^{d+1 \le j \le k+1} \underbrace{\begin{matrix} 1 & 0 \ldots \\ 1 & 1 \ldots \end{matrix}}_{\sigma_{\tilde{n}-j,m}}}_{\sigma_{\tilde{n},m} = \sum_{t=0}^{j-1} (-1)\omega_n^{-mt} + \theta_m \sigma_{\tilde{n}-j,m}}.$$

In fact, if the subsequences of length $\tilde{n}$ begin with one, then the mth trigonometric recurrence relation for $\sigma_{\tilde{n},m}$ can be obtained as

$$\sigma_{\tilde{n},m} = -\sum_{t=0}^{j-1} \omega_n^{-mt} + \theta_m \sigma_{\tilde{n}-j,m},$$

where the first term is the mth exponential sum of the prefix and $\sigma_{\tilde{n}-j,m}$ is the mth trigonometric recurrence relation corresponding to following subsequence; this subsequence also begins with one. The phase factor θ_m can be found by the shift theorem as follows: Before concatenating a prefix, for the first term z_1 of $\boldsymbol{z}$, we have

$$z_1 \omega_n^{-m0} = z_1.$$

Since there is just a one in the prefix, it follows that this prefix changes a sign of each term in the following subsequence. Therefore, after concatenating a prefix of length j, for the same term we obtain

$$\theta_m z_1 = -z_{j+1} \omega_n^{-mj}.$$

Since z_1 corresponds to x_1 and z_{j+1} corresponds to x_{j+1}, where x_1 and x_{j+1} are the same term, it follows that

$$\theta_m = -\omega_n^{-mj}.$$

Therefore

$$\sigma_{\tilde{n},m} = -\sum_{t=0}^{j-1} \omega_n^{-mt} - \omega_n^{-mj} \sigma_{\tilde{n}-j,m}, \tag{12}$$

Consider a finite geometric series in this equation; then

$$\sum_{t=0}^{j-1} \omega_n^{-mt} = \frac{1 - \omega_n^{-mj}}{1 - \omega_n^{-m}}$$
$$= \omega_n^{-m(j-1)/2} \frac{\sin\left(\frac{mj}{n}\pi\right)}{\sin\left(\frac{m}{n}\pi\right)},$$

and substituting it for the sum in (12), we get

$$\omega_n^{-mj} \sigma_{\tilde{n}-j,m} = -\sigma_{\tilde{n}} - \omega_n^{-m(j-1)/2} \frac{\sin\left(\frac{mj}{n}\pi\right)}{\sin\left(\frac{m}{n}\pi\right)}.$$

Multiplying both sides by ω_n^{mj}, we obtain (3).

The other case is when a leading series of zeros is running. First, we also consider the additional condition. In the case of $\tilde{n} \le \min(l, r)$, the leading run of zeros is the trailing one. This also gives us the only allowed sequence which length lies in the interval $[0, \min(l, r)]$.

$$\underbrace{\overbrace{\begin{matrix}0 \ldots 0\\ \scriptstyle 1 \ldots 1\end{matrix}}^{0 \le \tilde{n} \le \min(l,r)}}_{\sigma_{\tilde{n},m} = \sum_{t=0}^{\tilde{n}-1} \omega_n^{-mt}}.$$

Using (10) we get

$$\begin{aligned}\sigma_{\tilde{n}} &= \frac{1 - \omega_n^{-m\tilde{n}}}{1 - \omega_n^{-m}}\\ &= \omega_n^{-m(\tilde{n}-1)/2} \frac{\sin\left(\frac{m\tilde{n}}{n}\pi\right)}{\sin\left(\frac{m}{n}\pi\right)}.\end{aligned}$$

Substituting it for $\sigma_{\tilde{n},m}$ in (9), we obtain the condition in (8).

In the case of nonzero weight, there exist only zero weight prefixes which length lies in the interval $[0, l]$. Subsequences beginning with one follow the prefixes; therefore, we can consider a concatenation of this prefix and the following subsequences as

$$\underbrace{\overbrace{\begin{matrix}0 \ldots 0\\ \scriptstyle 1 \ldots 1\end{matrix}}^{0 \le j \le l} \underbrace{\begin{matrix}1 \;\; 0 \ldots\\ \scriptstyle -1 \, -1 \ldots\end{matrix}}_{\hat{\sigma}_{\tilde{n}-j,m}}}_{\sigma_{\tilde{n},m} = \sum_{t=0}^{j-1} \omega_n^{-mt} + \theta_m \hat{\sigma}_{\tilde{n}-j,m}}.$$

Thus, the mth expression for $\sigma_{\tilde{n},m}$ is

$$\sigma_{\tilde{n},m} = \sum_{t=0}^{j-1} \omega_n^{-mt} + \theta_m \hat{\sigma}_{\tilde{n}-j,m}.$$

Recall the reasoning that led us to (3). Arguing as there, and taking into account that the prefix now consists of all zeros, we get $\theta_m = \omega_n^{-mj}$ and

$$-\omega_n^{-mj} \hat{\sigma}_{\tilde{n}-j,m} = -\sigma_{\tilde{n},m} - \omega_n^{-m(j-1)/2} \frac{\sin\left(\frac{mj}{n}\pi\right)}{\sin\left(\frac{m}{n}\pi\right)}.$$

Multiplying both sides by $-\omega_n^{mj}$, we obtain (7). $\square$

Now consider a set $\boldsymbol{A}_{\boldsymbol{\sigma}_{\tilde{n}},\boldsymbol{\rho}_1,\boldsymbol{\rho}_2}$ of the rings $\mathscr{A}_{\sigma_{\tilde{n},0},\rho_{1,0},\rho_{2,0}}, \mathscr{A}_{\sigma_{\tilde{n},1},\rho_{1,1},\rho_{2,1}}, \ldots, \mathscr{A}_{\sigma_{\tilde{n},n-1},\rho_{1,n-1},\rho_{2,n-1}}$. Let $\boldsymbol{\sigma}_{\tilde{n}} = (\sigma_{\tilde{n},0}, \sigma_{\tilde{n},1}, \ldots, \sigma_{\tilde{n},n-1}) \in \mathbb{C}^n$ be a vector of centres of these rings, $\boldsymbol{\rho}_1 = (\rho_{1,0}, \rho_{1,1}, \ldots, \rho_{1,n-1}) \in \mathbb{R}^n$ and $\boldsymbol{\rho}_2 = (\rho_{2,0}, \rho_{2,1}, \ldots, \rho_{2,n-1}) \in \mathbb{R}^n$ vectors of theirs inner and outer radii.

Then consider those $\tilde{n}$-length subsequences from $\mathscr{S}(n)$ whose vectors of the exponential sums belong to $\boldsymbol{A}_{\boldsymbol{\sigma}_{\tilde{n}},\boldsymbol{\rho}_1,\boldsymbol{\rho}_2}$.

Let $C_{n,\tilde{n}}^{\boldsymbol{\sigma}_{\tilde{n}},\boldsymbol{\rho}_1,\boldsymbol{\rho}_2}$ and $\widehat{C}_{n,\tilde{n}}^{\boldsymbol{\sigma}_{\tilde{n}},\boldsymbol{\rho}_1,\boldsymbol{\rho}_2}$ be the number of these sequences, which begin with one and with a leading run of zeros respectively.

As above, under $C_{n,\tilde{n}}^{\boldsymbol{\sigma}_{\tilde{n}},\boldsymbol{\rho}_1,\boldsymbol{\rho}_2}$ we imply $C_{n,\tilde{n}}^{\boldsymbol{\sigma}_{\tilde{n}},\boldsymbol{\rho}_1,\boldsymbol{\rho}_2}(d, k, r)$ and under $\widehat{C}_{n,\tilde{n}}^{\boldsymbol{\sigma}_{\tilde{n}},\boldsymbol{\rho}_1,\boldsymbol{\rho}_2}$ we similarly imply $\widehat{C}_{n,\tilde{n}}^{\boldsymbol{\sigma}_{\tilde{n}},\boldsymbol{\rho}_1,\boldsymbol{\rho}_2}(d, k, l, r)$.

We state without proof the following proposition.

Proposition 2. *The numbers* $C_{n,\tilde{n}}^{\boldsymbol{\sigma}_{\tilde{n}},\boldsymbol{\rho}_1,\boldsymbol{\rho}_2}$ *and* $\widehat{C}_{n,\tilde{n}}^{\boldsymbol{\sigma}_{\tilde{n}},\boldsymbol{\rho}_1,\boldsymbol{\rho}_2}$ *can be expressed as:*

$$C_{n,\tilde{n}}^{\boldsymbol{\sigma}_{\tilde{n}},\boldsymbol{\rho}_1,\boldsymbol{\rho}_2} = \begin{cases} a_{\tilde{n}}, & \tilde{n} < d+1,\\ \sum_{j=d+1}^{\min(\tilde{n},k+1)} C_{n,\tilde{n}-j}^{\boldsymbol{\sigma}_{\tilde{n}-j},\boldsymbol{\rho}_1,\boldsymbol{\rho}_2} + b_{\tilde{n}}, & d+1 \le \tilde{n},\end{cases} \tag{13}$$

where components $\sigma_{\tilde{n}-j,m}$ *of the vector* $\boldsymbol{\sigma}_{\tilde{n}-j} = (\sigma_{\tilde{n}-j,0}, \sigma_{\tilde{n}-j,1}, \ldots, \sigma_{\tilde{n}-j,n-1})$ *are defined by* (3)*. Here initial condition*

$$a_{\tilde{n}} = \begin{cases} 1, & \tilde{n} \le r+1 \text{ and } \delta \textit{ is true},\\ 0, & \textit{otherwise},\end{cases} \tag{14}$$

and additional condition

$$b_{\tilde{n}} = \begin{cases} -1, & r+1 < \tilde{n} \le k+1 \text{ and } \delta \textit{ is true},\\ 1, & k+1 < \tilde{n} \le r+1 \text{ and } \delta \textit{ is true},\\ 0, & \textit{otherwise},\end{cases} \tag{15}$$

where the indicator function δ *is the logical conjunction of* n *statements each of which predicates of the* (m)*th exponential sum of trailing run* (6) *lies in the ring* $\mathscr{A}_{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}}$*, i.e.*

$$\delta = \bigwedge_{m=0}^{n-1} \left(\rho_{1,m} \le |\sigma_{\tilde{n},m} - \tilde{\sigma}_{\tilde{n},m}| \le \rho_{2,m}\right).$$

If a leading series is running, then

$$\widehat{C}_{n,\tilde{n}}^{\boldsymbol{\sigma}_{\tilde{n}},\boldsymbol{\rho}_1,\boldsymbol{\rho}_2} = \sum_{j=0}^{\min(\tilde{n},l)} C_{n,\tilde{n}-j}^{\hat{\boldsymbol{\sigma}}_{\tilde{n}-j},\boldsymbol{\rho}_1,\boldsymbol{\rho}_2} + \hat{b}_{\tilde{n}}, \tag{16}$$

where components $\hat{\sigma}_{\tilde{n}-j,m}$ *of the vector* $\hat{\boldsymbol{\sigma}}_{\tilde{n}-j} = (\hat{\sigma}_{\tilde{n}-j,0}, \hat{\sigma}_{\tilde{n}-j,1}, \ldots, \hat{\sigma}_{\tilde{n}-j,n-1})$ *are defined by* (7)*. Here additional condition*

$$\hat{b}_{\tilde{n}} = \begin{cases} -1, & r < \tilde{n} \le l \text{ and } \hat{\delta} \textit{ is true},\\ 0, & \textit{otherwise},\end{cases}$$

where the indicator function $\hat{\delta}$ *is the logical conjunction of* n *statements each of which predicates of the* (m)*th exponential sum of trailing run* $-\tilde{\sigma}_{\tilde{n},m}$ *lies in the ring* $\mathscr{A}_{\sigma_{\tilde{n},m},\rho_{1,m},\rho_{2,m}}$*, i.e.*

$$\hat{\delta} = \bigwedge_{m=0}^{n-1} \left(\rho_{1,m} \le |\sigma_{\tilde{n},m} + \tilde{\sigma}_{\tilde{n},m}| \le \rho_{2,m}\right).$$

IV. Algorithms for Encoding and Decoding Spectrum Shaped $dklr$ Sequences

Now let $\hat{\mathscr{S}}$ denotes a set of spectrum shaped binary run-length constrained sequences $\boldsymbol{x} = (x_1, x_2, \ldots, x_{\tilde{n}})$ of length $\tilde{n}$. Let the set $\hat{\mathscr{S}}$ be ordered lexicographically. From it follows that the lexicographic index $N(\boldsymbol{x}) \in \{\mathbb{N}, 0\}$ of $\boldsymbol{x} \in \hat{\mathscr{S}}$ is given by

$$N(\boldsymbol{x}) = \sum_{j=1}^{\tilde{n}} x_j W(\boldsymbol{p}),$$

where $W(\boldsymbol{p})$ denotes the number of sequences in $\hat{\mathscr{S}}$ with given prefix $\boldsymbol{p} = (x_1, x_2, \ldots, x_{j-1}, 0)$.

The decoding algorithm, for given sequence $\boldsymbol{x}$, find its lexicographic index $N(\boldsymbol{x}) < |\hat{\mathscr{S}}|$. This is done by successive approximation method using $W(\boldsymbol{p})$ as the weight of term x_j.

By $a_j(\boldsymbol{p})$ denote the number of trailing zeros of the prefix $\boldsymbol{p}$. By $\nu_{j-1} = \sum_{i=1}^{j-1} x_i$ denote the weight of this prefix. Since $\boldsymbol{p}$ is the prefix of $\boldsymbol{x}$, it follows that subsequence $\tilde{\boldsymbol{x}} = (x_j, x_{j+1}, \ldots, x_{\tilde{n}})$ is the rest of $\boldsymbol{x}$, and l_j is the leading run of zeros in this subsequence. We define l_j as the complement of $a_j(\boldsymbol{p})$ in l (for a leading run of zeros) or in k (for the other runs of zeros) as follows:

$$l_j = \begin{cases} l - a_j(\boldsymbol{p}), & \nu_{j-1} = 0, \\ k - a_j(\boldsymbol{p}), & \text{otherwise.} \end{cases}$$

Now we can compute the number of the spectrum shaped sequences $W_{\boldsymbol{\sigma}_{\tilde{n}}}(\boldsymbol{p})$ as

$$W(\boldsymbol{p}) = \begin{cases} \widehat{C}^{\boldsymbol{\sigma}_{\tilde{n}-j}, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2}_{n, \tilde{n}-j}(d, k, l_j, r) - \tilde{b}_{\tilde{n}}, & l_j \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

By $\tilde{b}_{\tilde{n}}$ we take into account a trailing run as follows:

$$\tilde{b}_{\tilde{n}} = \begin{cases} 1, & r - a_j(\boldsymbol{p}) < \tilde{n} - j \leq \min(l_j, r) \text{ and } \tilde{\delta} \text{ is true,} \\ 0, & \text{otherwise,} \end{cases}$$

where the indicator function $\tilde{\delta}$ is the logical conjunction of n statements each of which predicates of the mth exponential sum of the trailing run

$$\tilde{\sigma}_{\tilde{n}-j,m} = \omega_n^{-m(\tilde{n}-j-1)/2} \frac{\sin\left(\frac{m(\tilde{n}-j)}{n}\pi\right)}{\sin\left(\frac{m}{n}\pi\right)}$$

lies in the ring $\mathscr{A}_{\sigma_{\tilde{n}-j,m}, \rho_{1,m}, \rho_{2,m}}$, i.e.

$$\tilde{\delta} = \bigwedge_{m=0}^{n-1} \left(\rho_{1,m} \leq |\sigma_{\tilde{n}-j,m} - \tilde{\sigma}_{\tilde{n}-j,m}| \leq \rho_{2,m}\right),$$

where components $\sigma_{\tilde{n}-j,m}$ of the vector $\boldsymbol{\sigma}_{\tilde{n}-j} = (\sigma_{\tilde{n}-j,0}, \sigma_{\tilde{n}-j,1}, \ldots, \sigma_{\tilde{n}-j,n-1})$ are defined as

$$\sigma_{\tilde{n}-j,m} = (-1)^{\nu_{j-1}} \omega_n^{mj} (\sigma_{\tilde{n},m} - \sigma_{j-1,m}) - \omega_n^m.$$

We introduce the next variables: a, w, $\boldsymbol{c} = (c_0, c_1, \ldots, c_{n-1})$, which correspond to $a_j(\boldsymbol{p})$, ν_{j-1}, $\boldsymbol{\sigma}_{j-1}$.

$N := 0$; $a := 1$; $w := 0$; $\boldsymbol{c} := 0$;
for $j := 1$ **to** $\tilde{n}$ **do**
  Get $W(\boldsymbol{p})$;
  if $x_j = 1$ **then**
    $N := N + W(\boldsymbol{p})$; $a := 1$; $w := w + 1$;
  else
    $a := a + 1$;
  end if . . . else
  for $m := 0$ **to** $n - 1$ **do**
    $c_m := c_m + (-1)^w \omega_n^{-m(j-1)}$;
  end for
end for.

The encoding (inverse) algorithm, for given lexicographic index $N(\boldsymbol{x}) < |\hat{\mathscr{S}}|$, find the corresponding $\boldsymbol{x}$.

$a := 1$; $w := 0$; $\boldsymbol{c} := 0$;
for $j := 1$ **to** $\tilde{n}$ **do**
  Get $W(\boldsymbol{p})$;
  if $N \geq W(\boldsymbol{p})$ **then**
    $N := N - W(\boldsymbol{p})$; $x_j := 1$; $a := 1$; $w := w + 1$;
  else
    $x_j := 0$; $a := a + 1$;
  end if . . . else
  for $m := 0$ **to** $n - 1$ **do**
    $c_m := c_m + (-1)^w \omega_n^{-m(j-1)}$;
  end for
end for.

V. Conclusion

We presented an enumerative approach for constructing binary run-length limited sequences to satisfy spectral constraints. By considering the spectral components of the DFT in the complex plane, we got the possibility to use Cover's enumerative scheme for encoding these spectrum shaped sequences. First, we defined the sequences whose mth spectral components lie in a certain ring. Secondly, we proposed recurrence relations for calculating the number of such sequences. Then, we expanded our method for vectors of all spectral components. Finally, we suggested algorithms for encoding and decoding these sequences.

References

[1] K. A. S. Immink, *Codes for Mass Data Storage Systems*, 2nd ed. Eindhoven, The Netherlands: Shannon Foundation Publishers, 2004.

[2] K. A. S. Immink, P. H. Siegel, and J. K. Wolf, "Codes for digital recorders," *IEEE Trans. Inf. Theory*, vol. 44, no. 6, pp. 2260–2299, Oct 1998.

[3] V. Braun and K. A. S. Immink, "An enumerative coding technique for DC-free runlength-limited sequences," *IEEE Trans. Commun.*, vol. 48, no. 12, pp. 2024–2031, Dec. 2000.

[4] H. Nyquist, "Certain topics in telegraph transmission theory," *Trans. AIEE*, vol. 47, pp. 617–644, 1928.

[5] E. Gorog, "Redundant alphabets with desirable frequency spectrum properties," *IBM J. Res. Dev.*, vol. 12, no. 3, pp. 234–241, Jul. 1968.

[6] J. Franklin and J. Pierce, "Spectra and efficiency of binary codes without DC," *IEEE Trans. Commun.*, vol. COM-20, no. 6, pp. 1182–1184, Dec. 1972.

[7] G. L. Pierobon, "Codes for zero spectral density at zero frequency," *IEEE Trans. Inf. Theory*, vol. IT-30, no. 2, pp. 435–439, Mar. 1984.

[8] B. H. Marcus and P. H. Siegel, "On codes with spectral nulls at rational submultiples of the symbol frequency," *IEEE Trans. Inf. Theory*, vol. IT-33, no. 4, pp. 557–568, Jul. 1987.

[9] K. Norris and D. S. Bloomberg, "Channel capacity of charge-constrained run-length limited codes," *IEEE Trans. Magn.*, vol. Mag-17, no. 6, pp. 3452–3455, Nov. 1981.

[10] K. A. S. Immink, "Properties and constructions of binary channel codes," Ph.D. dissertation, Eindhoven University of Technology, The Netherlands, May 1985.

[11] P. I. Vasil'ev, "Block run-length-limited coding for digital magnetic recording," Ph.D. dissertation, LIAP, Leningrad, USSR, 1991, (in Russian).

[12] O. Kurmaev, "Weight and charge distribution of binary run-length limited codes," in *Proceedings of Tenth International Workshop on Algebraic and Combinatorial Coding Theory*, Zvenigorod, Russia, Sep. 3–9 2006, pp. 174–178.

[13] T. M. Cover, "Enumerative source coding," *IEEE Trans. Inf. Theory*, vol. IT-19, no. 1, pp. 73–77, Jan. 1973.

[14] G. C. Danielson and C. Lanczos, "Some improvements in practical Fourier analysis and their application to X-ray scattering from liquids," *J. Franklin Inst.*, vol. 233, pp. 365–380 and 435–452, Apr. 1942.